Main 340/341 POTW Compilation

Caleb Ju

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Contents

1 Preface

The questions are listed by their source and their (subjective) difficulty, out of 10. I take no credit from these problems, as they are borrowed from math competitions or puzzles posted in a Discord group. Unless a source or credit is given, solutions are developed on my own, though they are likely not to be the most "natural" or elegant approach.

2 Questions

 # 1 Unknown (3/10)

A frog begins at lily pad 1. At each step, it remains in place with probability 17/20 or jumps to the next lily pad (i.e., $i \to i+1$) with probability 3/20. Let d_n be the lily pad number after n steps, and let $E_n = \mathbb{E}[1/d_n]$. Find the least n such that $1/E_n > 2021$.

Hint: In general, $1/E_n \neq \mathbb{E}[d_n]$. Instead, use the definition of expectation. You may find the Binomial Theorem useful.

 # 2 Algorithms (Discrete Probability)

Given a coin which has heads with probability $0 < p < 1$ (which you don't know the value of), design a procedure to simulate a fair, i.e., $p = 1/2$, coin toss. Then find the expected runtime of the procedure.

 # 3 2019 IMOSL A5

Let $x_1, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{i=1}^{n} \prod_{j=1, j\neq i}^{n} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0: & \text{if } n \text{ is even} \\ 1: & \text{if } n \text{ is odd} \end{cases}.$$

 # 4 2021 AOPS Practice AMC 12 Q21

The number $1+2^{21}+4^{21}$ is divisible by exactly one two-digit prime p. What is the sum of the digits of p?

 # 5 2006 IMO Q4

Find all pairs (x, y) of integers such that

$$1+2^x+2^{2x+1}=y^2.$$

3 Solutions

1

The solution is $n = 13473$.

Let $p = 3/20$. First, we write the probability that the frog is at lilypad k as

$$\mathbb{P}(d_n = k) = p^{k-1}(1-p)^{n-(k-1)}\binom{n}{k-1},$$

which can be derived as by viewing the frog jumping $k-1$ and not jumping the remaining times. The combinations comes from choosing different $(k-1)$-sized subsets of times to jump at. By the law of the unconscious statistician, we want to find the smallest n where (we let the sum run from 1 to $n+1$ since we start at 1 and have n opportunities to jump)

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{d_n}\right] &= \sum_{k=1}^{n+1}\mathbb{P}(d_n = k)\cdot\frac{1}{k} \\
&= \sum_{k=1}^{n+1} p^{k-1}(1-p)^{(n+1)-k}\frac{n!}{((n+1)-k)!k!} \\
&= \frac{1}{(n+1)p}\sum_{k=1}^{n+1} p^{k}(1-p)^{(n+1)-k}\binom{n+1}{k} \\
&\overset{(1)}{=} \frac{1}{(n+1)p}\left[\underbrace{(p+(1-p))^{n+1}}_{1} - (1-p)^{n+1}\right] < \frac{1}{2021},
\end{aligned}$$

where (1) follows by the Binomial Theorem. Rearranging terms and plugging in $p = 3/20$, we derive the equivalent inequality

$$\frac{2021\cdot 20}{3}\left(1-(17/20)^{n+1}\right) < n+1. \tag{1}$$

The rest is just simple algebra and guess-and-check, which we write for completeness. If we assume $(1-p)^{n+1}\approx 0$, we can simplify and guess $n = 13473$, and it is easily verified that (1) holds. To verify this is the minimum such n, we show $n = 13472$ violates the inequality (indeed, there cannot exist a such smaller $n < 13472$ since from the the original problem, less times to jump means the average lilypad is smaller since we can never move backwards). First, we make the following observation,

$$(17/20)^{13473} < (0.9)^{10000} < 10^{-30} \text{ since } (0.9)^{1000} < 10^{-3}.$$

Therefore, using $2021\cdot 20/3 = 13743 + 1/3$, we have

$$\begin{aligned}
\left(13473+\frac{1}{3}\right)\left(1-(17/20)^{13473}\right) &> 13473 + \left(1/3 - 10^5\cdot(17/20)^{13473}\right) \\
&> 13473 + (1/3 - 10^{-25}) > 13473,
\end{aligned}$$

or that (1) is violated when $n = 13472$.

 # 2

See Professor Jeff Erickson's *Algorithms* textbook (Section 1.4.1).

 # 4

The prime is $p = 73$, and its sum is 10.

We can equivalently write the problem as finding a prime p such that

$$\begin{aligned} 1 + r + r^2 &\equiv 0 \pmod{p} \\ 128^3 &\equiv r \pmod{p}. \end{aligned}$$

Enumerating all primes and brute force checking all solutions, we find $p = 73$ satisfies these two properties, as

$$\begin{aligned} 128^3 &\equiv (-18)^3 \equiv (-18)(324) \equiv (-18)(32) \\ &\equiv (-6)(96) \equiv (-6)(23) \equiv -138 \equiv -65 \equiv 8 \pmod{73}. \end{aligned}$$

Thus, $r = 8$, and we have $1 + r + r^2 = 73$, which is what we needed to show.