Parallel Approximate Undirected Shortest Paths Via Low Hop Emulators

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Motivation for Parallel Algorithms

Assess the efficiency of parallel algorithms by analyzing:

- Depth, $D_P(n)$, is length of the longest series of operations
- Work, $W_P(n)$, is runtime in a sequential setting

We seek algorithms that are:

- Polylog(arithmic) depth , $D_P(n) = \mathcal{O}\big(\text{poly}(\log n)\big)$
- Nearly work-efficient, $W_P(n) = \mathcal{O}\big(T_S(n) \cdot \text{poly}(\log n)\big)$

Parallel algorithms for shortest paths problem

Given a (non–negative) weighted graph $G = (V, E)$ shortest paths problem finds the shortest path from $s \to t$ in G.

The single source shortest paths problem is the shortest path from $s \in S$ to every vertex in V.

Let $n = |V|$ and $m = |E|$. What algorithms are there?

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- Dijkstra's algorithm requires $\mathcal{O}(m + n \cdot \log(n))$ work/depth
- Bellman Ford requires $\mathcal{O}(nm)$ work and $\mathcal{O}(n)$ depth
- Floyd–Warshall requires $\mathcal{O}(n^3)$ work and $\mathcal{O}(\log n)$ depth

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- Floyd–Warshall requires $\mathcal{O}(n^3)$ work and $\mathcal{O}(\log n)$ depth

Question: Does there exist a $(1 + \varepsilon)$–shortest paths algorithm with $\mathcal{O}\big(m \cdot \text{poly}(\log n)\big)$ work and $\mathcal{O}(\text{poly}(\log n))$ depth?

Previous and New Results

Previous Results

1. Use hopset [Coh94] to solve $(1+\varepsilon)$–shortest paths in polylog depth and $\Omega(n^{2.1})$ work
2. Any $(1+\varepsilon)$–shortest paths algorithm with polylog depth using only hopsets requires $\Omega(n^{1+\varepsilon'})$ work [ABP18]

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This Paper

1. Construct a novel data structure, a low hop emulator, to solve approximate single source shortest paths

2. Compute $(1+\varepsilon)$–approximate shortest path in polylog depth and nearly linear work via [She17] and low hop emulators

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This Paper

1. Construct a novel data structure, a low hop emulator, to solve approximate single source shortest paths
2. Compute $(1+\varepsilon)$–approximate shortest path in polylog depth and nearly linear work via [She17] and low hop emulators

Given a graph $G = (V, E)$, a low hop emulator is a weighted graph $H = (V, F)$ where any shortest $(s-t)$–path with $\mathcal{O}(\log\log n)$ edge traversals and $|F| = \mathcal{O}\big(m \cdot \mathrm{poly}(\log n)\big)$.

Overview

1. Constructing a Low Hop Emulator
   - Constructing a subemulator
   - Recursive subemulators
   - Collapsing into a low–hop emulator
   - Constructing a low–hop emulator (in parallel)

2. $(1+\varepsilon)$ shortest paths in polylog depth and nearly linear work
   - Solving shortest paths via optimization and Sherman's framework

Constructing a low hop emulator

To construct a low hop emulator,

1. Construct a subemulator
2. Recursive subemulators
3. Collapse all subemulators down to a single graph $\rightarrow$ low–hop emulator

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A subemulator is a graph $H = (S, F')$ where $S \subset V$ and F' is a weighted edge set that approximates distances well.

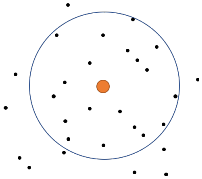
1. Every vertex $v \in V$ is close to a vertex in S
2. Distances in H approximate distances in G

Constructing a subemulator

A subemulator is a graph $H = (S, F')$ where $S \subset V$ and F' is a weighted edge set that approximates distances well.

1. Select vertices first by sampling

Our analysis depends on the ball $B_{G,b}(v)$, which is the closest b vertices (graph distance) to v in G,



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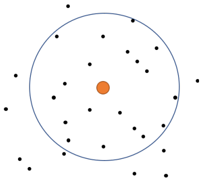

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1. Select vertices first by sampling
2. Add more vertices to ensure vertex in V is close to a vertex in S
3. Add edges and assign weights so that local distances are well–approximated

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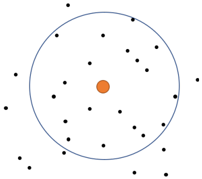


Selecting vertices

Fix a ball size b (to be defined later).

1. Construct S by sampling every vertex with probability $p = \min(50\frac{\log n}{b}, \frac{1}{2})$
2. If $v \in V$ is not near any vertex in S, add v to S
3. Store the leader $q(v) \leftarrow$ closest vertex $u \in S$ to $v \in V$

Output: A sparse vertex set S and mapping $q : V \to S$

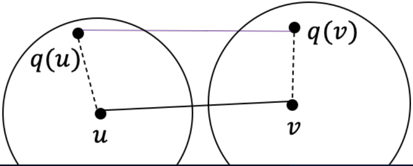
Adding edges

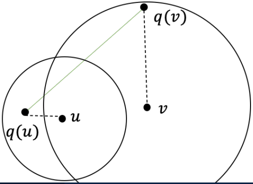
Adding edges:

1. For all $(u, v) \in E$, add edge $(q(u), q(v))$
2. Fix $v \in V$. $\forall u \in B(v)$, add edge $((q(u), q(v))$

When adding an edge $e = (q(u), q(v))$ to F, update weight

$$w(e) = \min \begin{cases} w(e), \text{(initialize to } \infty) \\ d_G(q(u), u) + d_G(u, v) + d_G(v, q(v)) \end{cases}.$$





Properties of subemulators

Given a subemulator $H = (S, F')$, Size

- $\mathbb{E}\big[|S|\big] < n$
- $|F'| \leq m + nb$

Distance approximation

- For any $u, v \in S$,

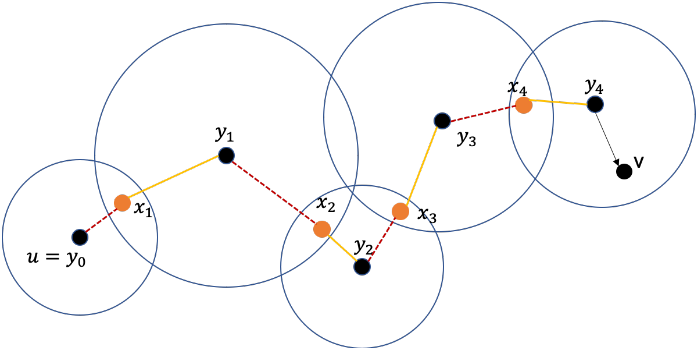
$$d_G(u, v) \leq d_H(u, v) \leq 8 \cdot d_G(u, v).$$

- For any $u, v \in V$,

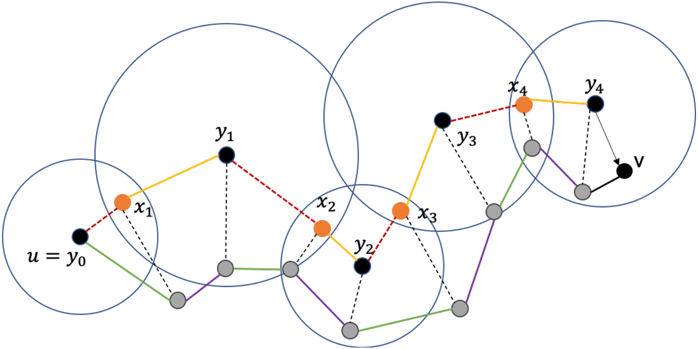
$$d_H\big(q(u), q(v)\big) \leq d_G\big(u, q(u)\big) + 22 \cdot d_G(u, v) + d_G\big(v, q(v)\big).$$

8-approximation for shortest (u, v)–path

Consider arbitrary $u, v \in S$ and its shortest path in G.



8-approximation for shortest (u, v)–path



$$
\begin{aligned}
d_H(u, v) &\leq \sum_{i=1}^{t} {\color{green} w_H\big(q(y_{i-1}), q(x_i)\big)} + {\color{purple} w_H\big(q(x_i), q(y_i)\big)} + w_H(y_t, v) \\
&\leq 8 \cdot \Big(\sum_{i=1}^{t} {\color{red} w_G(y_{i-1}, x_i)} + {\color{orange} w_G(x_i, y_i)} + w_G(y_t, v) \Big) \\
&= 8 \cdot d_G(u, v).
\end{aligned}
$$

Constructing a Low Hop Emulator

High level approach:

1. Construct a subemulator
2. Recursive subemulators
3. Collapse all subemulators down to a single graph $\rightarrow$ low–hop emulator

Recursive subemulators

Set $b_0 \ll n$.

Let the first subemulator H_0 be the original graph G.

While $|V(H_i)| \geq b_i$:

1. $\forall v \in V(H_i)$, save $B_{H_i,b_i}(v)$
2. $H_{i+1} \leftarrow \text{SUBEMULATOR}(H_i, b_i)$
3. Update $b_{i+1} \leftarrow b_i^{1.25}$ and $i \leftarrow i+1$

Output: Set of subemulators H_i and set of balls $B(v'), \forall v' \in H_i$

Recursive subemulators

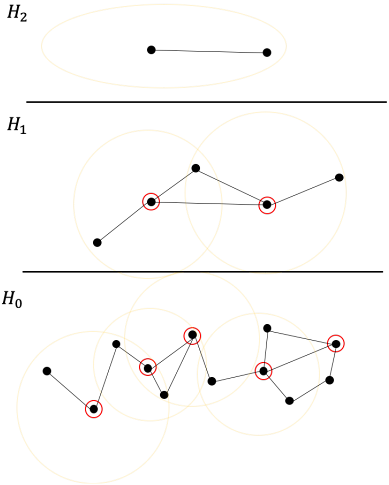


Figure: Recursive subemulators

Calculating distances from multiple subemulators

Let u, v be arbitrary vertices whose distance we want to compute.

$d = 0,\ i = 0$
$u_0 = u,\ v_0 = v$
While u_i, v_i are not in the same ball:

1. $d \leftarrow d + d_{H_i}\big(q(u_i), u_i)\big) + d_{H_i}\big(v, q(v_i)\big)$
2. $u_{i+1} \leftarrow q(u_i),\ v_{i+1} \leftarrow q(v_i)$
3. $i \leftarrow i + 1$

Return $d + d_{H_i}(u_i, v_i)$.

Calculating distances from multiple subemulators

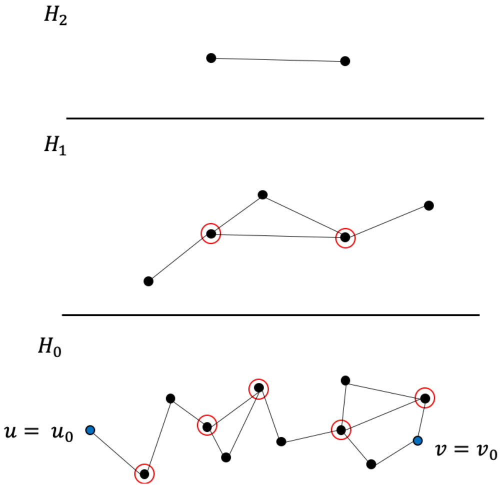


Figure: Traversing H_0

Calculating distances from multiple subemulators

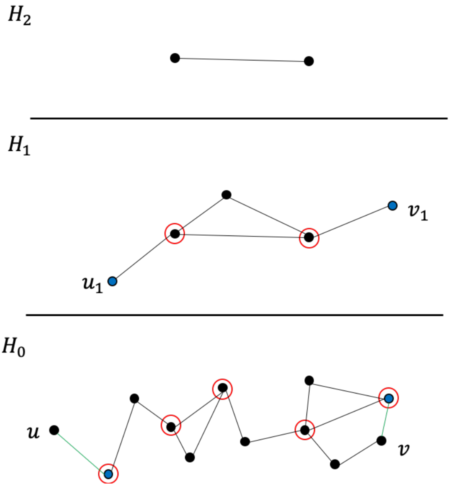


Figure: Traversing H_1

Calculating distances from multiple subemulators

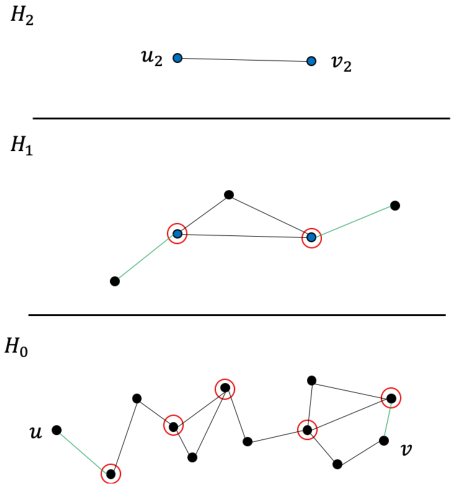


Figure: Traversing H_2

Calculating distances from multiple subemulators

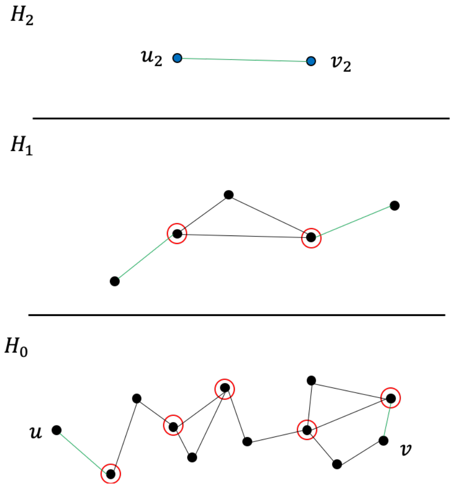


Figure: Near vertices exact distance

Properties of a distance oracle

1. Number of subemulators is $\mathcal{O}(\log\log n)$
2. Traversing up–down subemulators to get (u, v)–shortest path is $26^{\mathcal{O}(\log\log n)} = \text{poly}(\log n)$–approximation
3. Store $\mathcal{O}(\log\log n)$ subemulators and each vertex's ball

Constructing a Low Hop Emulator

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Collapsing towards a low–hop emulator

1. $V' \leftarrow \big(V(H_0), 0\big) \bigcup \ldots \big(V(H_i), i\big) \ldots \bigcup \big(V(H_t), t\big)$
2. Add the edge from a vertex up to its leaders in next level
3. Keep all edges within each subemulator
4. Add the edge between close vertices that are within same subemulator

Let t be number of levels. Set

$$w_F\big((u,i),(v,j)\big) = 27^{t-\max(i,j)} \cdot d_{H_i}(u,v).$$

Collapsing towards a low–hop emulator

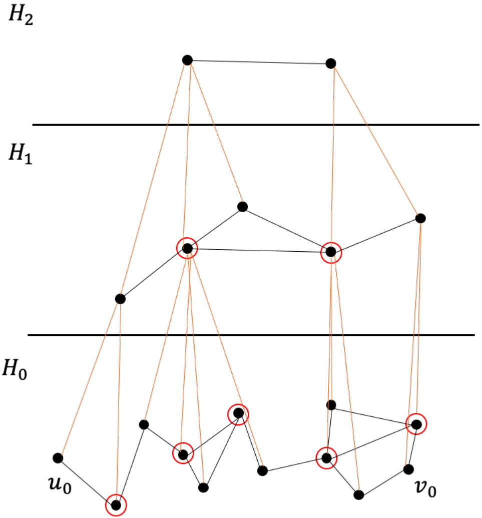

Figure: Edges between levels

Collapsing towards a low–hop emulator

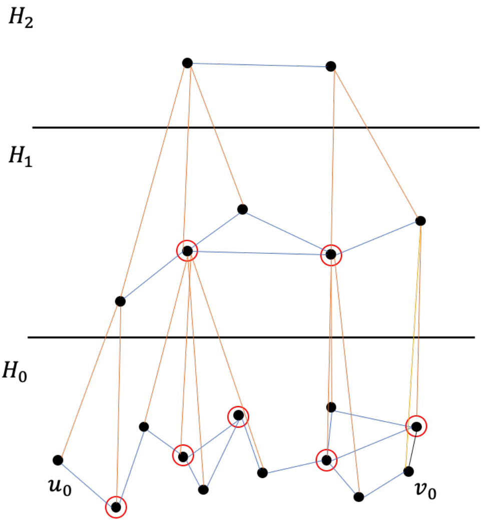

Figure: Edges within a level

Collapsing towards a low–hop emulator

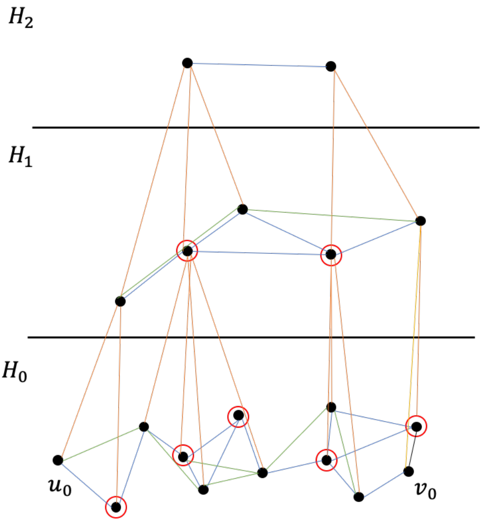

Figure: Edges between nearby vertices

Collapsing towards a low–hop emulator

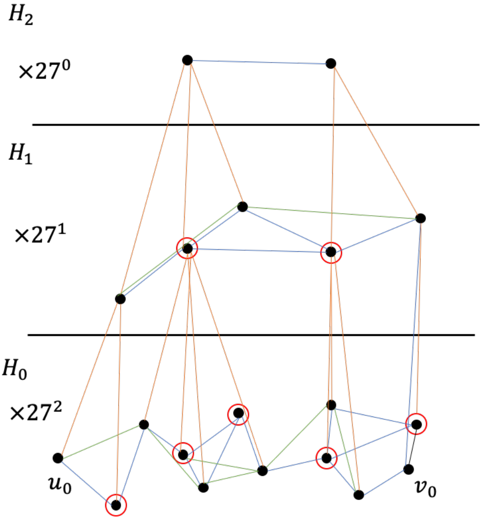

Figure: Scaling edges

Collapsing towards a low–hop emulator

1. Can remove copies of a vertex v, $(v, 0), (v, 1), \ldots$, and preserve edges by squishing all copies of vertex to bottom level

   OK since have orange edges between (v, i) and $(v, i + 1)$ are weight zero.

2. There always exists a shortest $(u - v)$–path with minimal edge traversals without using blue edges

   Close edges will be covered by green edges and far edges will never be used.

Collapsing towards a low–hop emulator

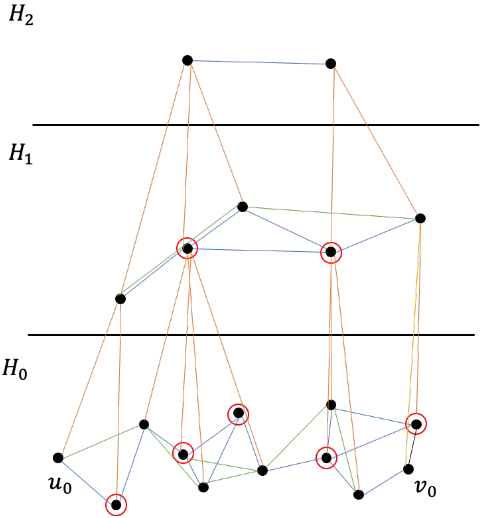

Figure: Almost low–hop emualtor

Collapsing towards a low–hop emulator

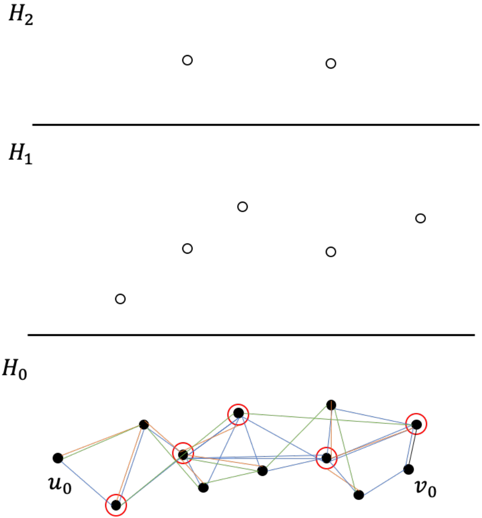

Figure: Removing redundant vertices

Collapsing towards a low–hop emulator

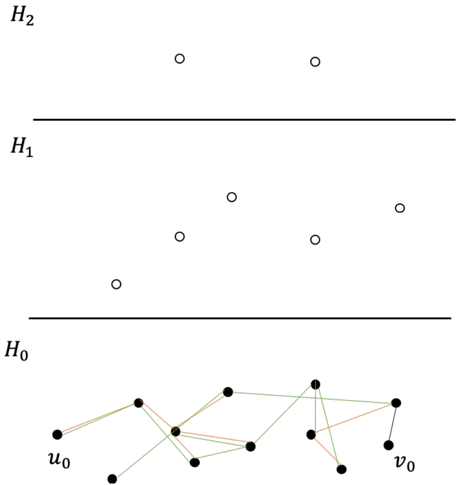

Figure: Remove blue edges

Recap

High level approach:

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Recap

High level approach:

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Our constructed graph $H = (V', F)$:

1. Shortest path requires $\mathcal{O}(\log\log n)$ traversals
2. Distance is $27^{\mathcal{O}(\log\log n)} = \text{poly}(\log n)$–approximated
3. $V' = V$
4. $\mathbb{E}[|F|] = \mathcal{O}\big(n \cdot \text{poly}(\log n)\big)$

Corollary: H is a low–hop emulator.

Constructing a low–hop emulator (in parallel)

Assume that we have computer that allows concurrent reads and concurrent writes (CRCW).

1. Create the ball centered at v
2. Selecting vertices
3. Adding edges
4. Recursive subemulators
5. Collapsing towards a low–hop emulator

Creating the ball centered at v (in parallel)

Uses path–doubling idea as done in Floyd–Warshall.

$L_0(v) \leftarrow$ closest b neighbors of v (including v itself)
For $u \in L_0(v)$, compute $\text{dist}^{(0)}(v, u)$

For $i = 1, \ldots, t = \lceil \log n \rceil$:

1. $\forall v, u \in V$, set $\text{dist}^{(i)}(v, u) \leftarrow \infty$
2. Fix $v \in V$. Consider (v, x, u) such that $x \in L_{i-1}(v)$ and $u \in L_{i-1}(x)$. Update $\text{dist}^{(i)}(v, u)$ if $\text{dist}^{(i-1)}(v, x) + \text{dist}^{(i-1)}(x, u)$ is smaller
3. For every $v \in V$, set $L_i(v) \leftarrow$ closest b vertices

Output: $B(v) \leftarrow L_t$ and exact distance $d^{(t)}(v, \cdot)$

Creating the ball centered at v (in parallel)

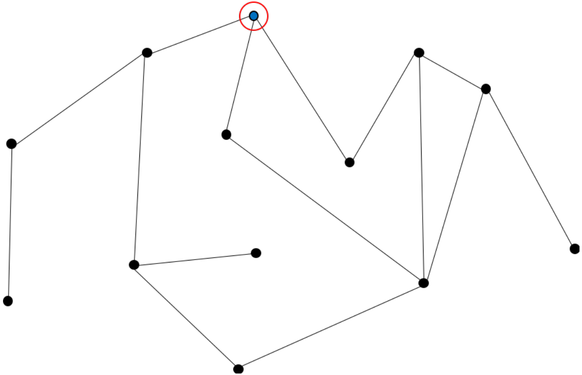

Figure: Expanding ball with $b = 5$. Reach of 0

Creating the ball centered at v (in parallel)

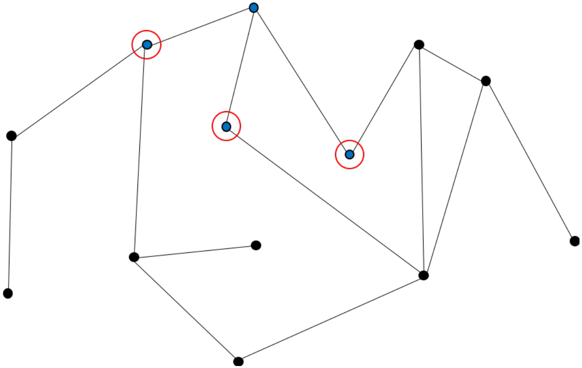

Figure: Expanding ball with $b = 5$. Reach of 1

Creating the ball centered at v (in parallel)

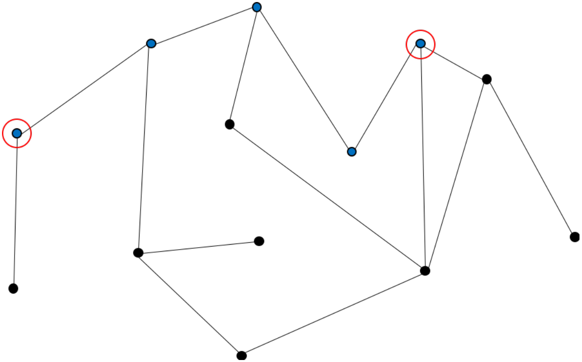

Figure: Expanding ball with $b = 5$. Reach of 2

Creating the ball centered at v (in parallel)

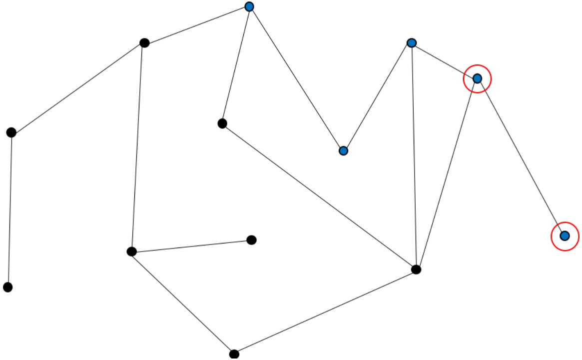

Figure: Expanding ball with $b = 5$. Reach of 4

Creating the ball centered at v (in parallel)

$L_0(v) \leftarrow$ closest b neighbors of v. Compute $\mathrm{dist}^{(0)}(v, u)$

- Use polylog depth and nearly work-efficient parallel sorting algorithm $\Rightarrow \mathcal{O}(\log^2 n)$ depth and $\mathcal{O}(m \log^2 n)$ work
- Initialize distance in $\mathcal{O}(1)$ depth and $\mathcal{O}(m)$ work

Creating the ball centered at v (in parallel)

For $i = 1, \ldots, t = \lceil \log n \rceil$:

1. $\forall v, u \in V$, set $\text{dist}^{(i)}(v, u) \leftarrow \infty$
2. Fix $v \in V$. Consider (v, x, u) such that $x \in L_{i-1}(v)$ and $u \in L_{i-1}(x)$. Update $\text{dist}^{(i)}(v, u)$ if ...
3. For every $v \in V$, set $L_i(v) \leftarrow$ closest b vertices

We perform ball computations on each vertex in G or a subemulator H_i. Repeat $\log n$ times:

1. $\leq b^2$ vertex pairs per vertex $v \in V$.
2. $\mathbb{E}[|V(H_i)|] = \mathcal{O}(n \cdot \text{poly}(\log n)) / b_i^2 \Rightarrow \mathcal{O}(n \cdot \text{poly}(\log n))$ work and $\mathcal{O}(1)$ depth per iteration
3. Use sorting algorithm

Selecting vertices (in parallel)

Fix a ball size b (to be defined later).

1. Construct S by sampling every vertex
2. If $v \in V$ is not near any vertex in S, add v to S
3. Store the leader $q(v) \leftarrow$ closest vertex $u \in S$ to $v \in V$

Selecting vertices (in parallel)

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1. $\mathcal{O}(1)$ depth and $\mathcal{O}(n)$ work
2. Construct a ball. Check requires $\mathcal{O}(1)$ depth and $\mathcal{O}(n)$ work
3. Ditto

Adding edges (in parallel)

Adding edges:

1. For all $(u, v) \in E$, add edge $\big(q(u), q(v)\big)$
2. Fix $v \in V$. $\forall u \in B(v)$, add edge $\big((q(u), q(v)\big)$

When adding an edge $e = \big(q(u), q(v)\big)$ to F, update weight

$$w(e) = \min \begin{cases} w(e), (\text{initialize to } \infty) \\ d_G(q(u), u) + d_G(u, v) + d_G(v, q(v)) \end{cases}.$$

Adding edges (in parallel)

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$$w(e) = \min \begin{cases} w(e), (\text{initialize to } \infty) \\ d_G(q(u), u) + d_G(u, v) + d_G(v, q(v)) \end{cases} .$$

1. Uses ball to compute exact distances. $\mathcal{O}(1)$ depth and $\mathcal{O}(m)$ work
2. Ditto
3. Ditto

Recursive subemulators (in parallel)

Set $b_0 \ll n$.

Let the first subemulator H_0 be the original graph G.

While $|V(H_i)| \geq b_i$:

1. $\forall v \in V(H_i)$, save $B_{H_i, b_i}(v)$
2. $H_{i+1} \leftarrow \text{SUBEMULATOR}(H_i, b_i)$
3. Update $b_{i+1} \leftarrow b_i^{1.25}$ and $i \leftarrow i + 1$

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Repeat $\mathcal{O}(\log \log n)$ times:

1. Ball construction
2. Proved is polylog depth and nearly linear
3. $\mathcal{O}(1)$

Collapsing towards a low–hop emulator (in parallel)

For every subemulator $H_0, \ldots, H_{\mathcal{O}(\log logn)}$:

1. Add the edge from a vertex up to its leaders in next level
2. Add the edge between close vertices that are within same subemulator

Let t be number of levels. Set

$$w_F(u, v) = 27^{t - \max(i,j)} \cdot d_{H_i}(u, v).$$

Collapsing towards a low–hop emulator (in parallel)

For every subemulator $H_0, \ldots, H_{\mathcal{O}(\log\, log n)}$:

1. Add the edge from a vertex up to its leaders in next level
2. Add the edge between close vertices that are within same subemulator

Let t be number of levels. Set

$$w_F(u, v) = 27^{t-\max(i,j)} \cdot d_{H_i}(u, v).$$

Repeat $\mathcal{O}(\log \log n)$ for each subemulator

1. $\mathcal{O}(1)$ depth and $\mathcal{O}\big(n \cdot \mathrm{poly}(\log n)\big)$ work
2. Ditto
3. Weights: Ditto

Constructing a low–hop emulator (in parallel)

Recap. Assuming we have a concurrent read, concurrent write computational model, we can solve the following problems in polylog depth and near linear work:

1. Create the ball centered at v
2. Selecting vertices
3. Adding edges
4. Recursive subemulators
5. Collapsing towards a low–hop emulator

Constructing a low–hop emulator can be done in same depth and work.

Applications

Compute the following problems with polylog approx, polylog depth, and nearly linear work:

1. SSSP via Bellman-Ford
2. Bourgain's embedding
3. Low diameter decomposition
4. Metric tree embedding

Uncapacitated min–cost flow (transshipment) problem

Let $W \in \mathbb{R}^{m \times m}$ be a diagonal matrix of weights. Let $A \in \mathbb{R}^{n \times m}$ be the incidence matrix,

$$A_{iu} = \begin{cases} 1 \ : \ \exists \text{ edge } u = (i,j) \\ -1 : \ \exists \text{ edge } u = (j,i) \\ 0 \ : \ \text{otherwise} \end{cases} .$$

Find a vector $f \in \mathbb{R}^m$ such that

$$\min_{f \in \mathbb{R}^m} \|Wf\|_1$$
$$\text{s.t. } Af = b,$$

where $b \in \mathbb{R}^n$ is the demand vector, where we require $\sum_i b_i = 0$.

If $b(s) = 1, \ b(t) = -1$, then solves (s,t)–shortest path length.

Uncapacitated min–cost flow problem

An equivalent problem:

Let $x = Wf$. Find the optimal x^* such that

$$x^* = \min_{x\mathbb{R}^m} \|x\|_1$$
$$\text{s.t. } AW^{-1}x = b.$$

Uncapacitated min–cost flow problem

An equivalent problem:

Let $x = Wf$. Find the optimal x^* such that

$$x^* = \min_{x\mathbb{R}^m} \|x\|_1$$
$$\text{s.t. } AW^{-1}x = b.$$

Lemma: There exists $(1+\varepsilon)$–approximation algorithm to the optimization problem above in that runs in polylog depth if there exists a matrix P such that

$$\|x^*\|_1 \leq \|Pb\|_1 \leq \mathcal{O}(\operatorname{poly}\log n) \cdot \|x^*\|_1 .$$

Earth Movers Distance Problem

The Earth Mover's Distance (EMD) problem is

$$\min_{\pi: V\times V \to \mathbb{R}_{\geq 0}} \sum_{(u,v)\in V\times V} \pi(u,v)\cdot \|\phi(u)-\phi(v)\|_1$$

$$\text{s.t. } \forall u \in V, \sum_{v\in V}\pi(u,v) - \sum_{v\in V}\pi(v,u) = b_u.$$

Find a vector $f \in \mathbb{R}^m$ such that

$$\min_{f\in\mathbb{R}^m} \|Wf\|_1$$

$$\text{s.t. } Af = b,$$

Bourgain's Embedding

Theorem (Bourgain's Embedding)

Every metric space (V, d_V) can be embedded in ℓ_p with distortion $\mathcal{O}(\log n)$.

Given a graph $G = (V, E)$ and distance $d : V \times V \to \mathbb{R}^+$, there exists a mapping $\phi : V \to [\Delta]^{\mathcal{O}(\log^2 n)}$ such that

$$d(u, v) \leq \|\phi(u) - \phi(v)\|_1 \leq \mathcal{O}(\log n) d(u, v),$$

where $\Delta \leq \sum_{e \in E} w_e$.

Using Bourgain's embedding via low–hop emulators, we can find a mapping $\phi : V \to [\Delta]^\eta$ such that

$$\mathrm{OPT}_{EMD}(b) \leq \mathcal{O}(\mathrm{poly}(\log n))\mathrm{OPT}_{transshipment}(b).$$

Creating a preconditioner using grids

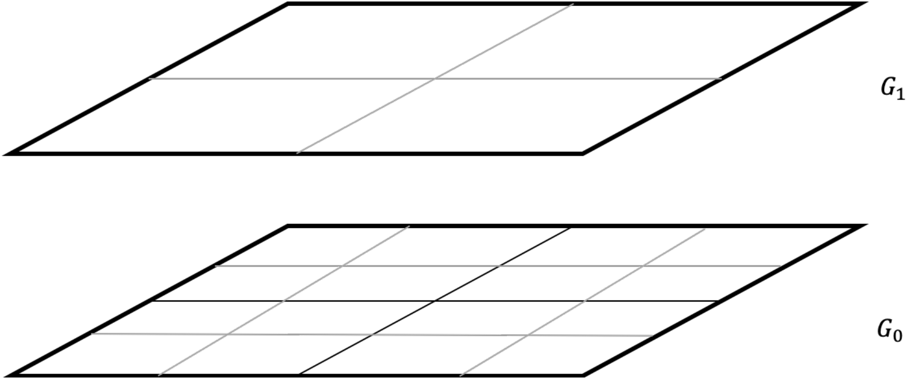

Figure: Grids

Creating a preconditioner

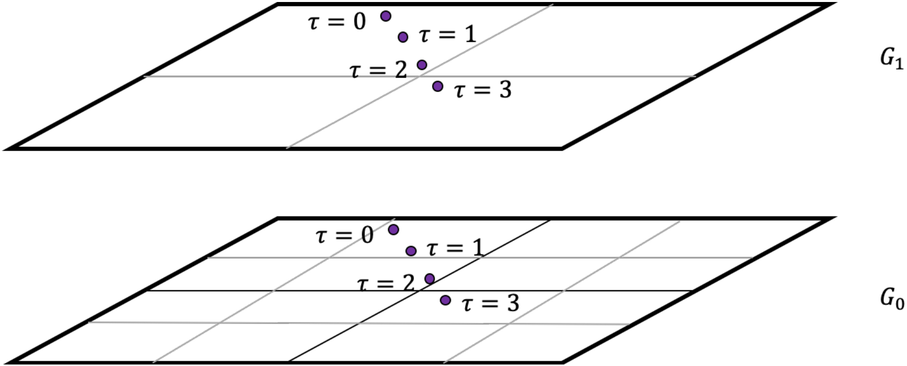

Figure: Grids with points

Creating a preconditioner

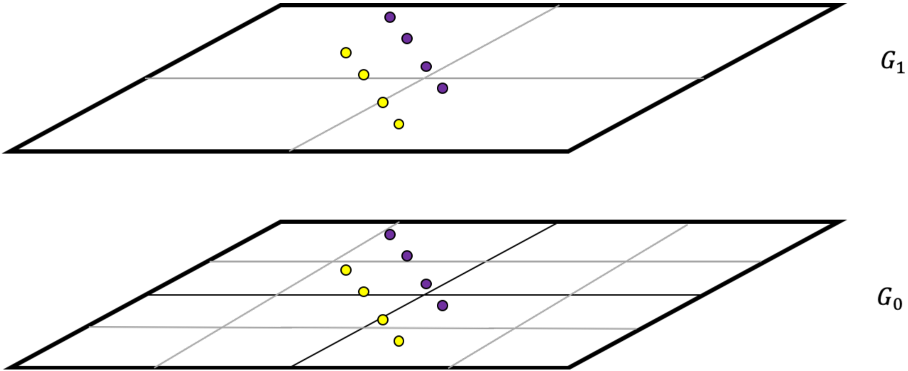

Figure: Points in same cell vs. random shift τ

Creating a preconditioner

Construct a series of $L = 1 + \log \Delta$ grids G_i as above.

Let $\tau \in [\Delta]$ be a uniform random variable.

Lemma: Define the vector

$$h_{(i,C)} = \eta \cdot 2^i \sum_{v \in V : \phi(v) + \tau \cdot \mathbf{1}_\eta \in C} b_v,$$

where i is the level of the grid (ie. G_i) and $C \in G_i$ is a cell. Then,

1. $\mathbb{E}_\tau[\|h\|_1] \leq 2L\eta \cdot \mathrm{OPT}_{EMD}(b)$
2. $\|h\|_1 \geq \mathrm{OPT}_{EMD}(b)$,

where $\eta = \mathcal{O}(\log^2 n)$.

Creating a preconditioner

Construct a vector h' such that $\mathbb{E}\left[\,\|h\|_1\,\right] = \|h'\|_1$,

$$h'_{(i,C,\tau)} = \eta \cdot \sum_{v \in V : \phi(v) + \tau \cdot \mathbf{1}_\eta \in C} b_v .$$

Creating a preconditioner

Construct a vector h' such that $\mathbb{E}\left[\,\|h\|_1\,\right] = \|h'\|_1$,

$$h'_{(i,C,\tau)} = \eta \cdot \sum_{v \in V : \phi(v) + \tau \cdot \mathbf{1}_\eta \in C} b_v .$$

Prescribes a matrix P' where

$$P'_{(i,C,\tau),v} = \begin{cases} \eta \ : \ \phi(v) + \tau \cdot 1_\eta \in C \\ 0 \ : \ \text{otherwise} \end{cases} .$$

Then $h' = P'b$, and

$$\|x^*\|_1 \leq \left\|h'\right\|_1 = \left\|P'b\right\|_1 \leq (\text{poly}(\log n)) \, \|x^*\|_1$$

Creating a preconditioner

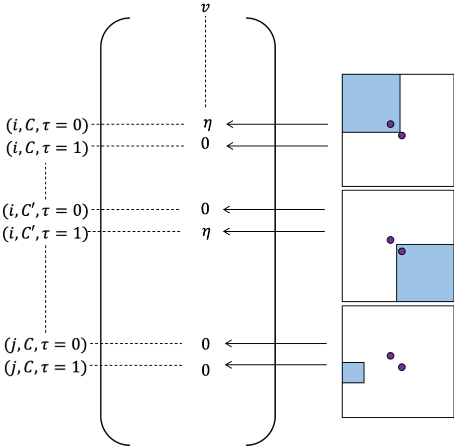

Figure: Preconditioner vs. Grids

Uncapacitated min–cost flow problem

We want to solve the following problem:

Let $x = Wf$. Find the optimal x^* such that

$$x^* = \min_{x\mathbb{R}^m} \|x\|_1$$
$$\text{s.t. } AW^{-1}x = b.$$

Uncapacitated min–cost flow problem

We want to solve the following problem:

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$$\text{s.t. } AW^{-1}x = b.$$

Lemma: There exists $(1+\varepsilon)$–approximation algorithm to the optimization problem above in that runs in polylog depth using a matrix P' where

$$\|x^*\|_1 \leq \|P'b\|_1 \leq \mathcal{O}(\text{poly}\log n) \cdot \|x^*\|_1 .$$

Summary

1. Constructing a Low Hop Emulator
   - Constructing a subemulator
   - Recursive subemulators
   - Collapsing into a low–hop emulator
   - Constructing a low–hop emulator (in parallel)

2. $(1+\varepsilon)$ shortest paths in polylog depth and nearly linear work
   - Solving shortest paths via optimization and Sherman's framework

Thanks. Questions?